SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2007
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #2
(Year 12 Half Yearly)

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question.
- Hand in your answers in 3 separate bundles. Section A (Question 1), Section B (Question 2) and Section C (Question 3)

Total Marks – 60

- Attempt questions 1-3
- All questions are **NOT** of equal value.

Examiner: *A Ward*

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Section A – Start a new booklet.

Question 1 (20 marks). **Marks**

a) The n^{th} term of an arithmetic series is $12-4n$. Find the first term, *a* and the common difference, *d*. 2

b) Differentiate with respect to *x*

(i) $\cos 2x$ 1

(ii) e^{x^2+1} 1

(iii) $\ln x^5$ 1

(iv) $\dfrac{\sin x}{x^2}$ 2

c) Convert $\dfrac{4\pi}{3}$ radians to degrees. 1

d) Express 210° in radian measure. 1

e) Find $\cos\theta$ and $\sin\theta$, if $\tan\theta = \dfrac{-7}{24}$ and θ is reflex. 2

f) Find:

(i) $\int x^{2n}\,dx$ where $n \neq -\dfrac{1}{2}$ 1

(ii) $\int \sec^2(2x+3)\,dx$ 1

g) Find the gradient of the curve $y=(x-3)(x^2+2)$ at the point (1,-6) 2

h) Find the length of the arc of a circle of radius $4cm$ which subtends an angle of $\dfrac{\pi}{3}$ radians at the centre. 1

i) Find the equation of the tangent to the curve $y = x^2 - 3x + 2$ at the point where it cuts the y-axis. 1

j) The co-ordinates of the vertices A, B and C of the triangle ABC are (-3,7), (2,19) and (10,7) respectively. Prove that the triangle is isosceles. 3

End of Section A.

Section B – Start a new booklet.

Question 2 (20 marks). **Marks**

a) The gradient function of a curve is $3x^2 - 5x + 1$ and the curve itself passes through (0,3). Find the equation of the curve. 2

b) Find:

(i) $\int e^{3x+2} dx$ 1

(ii) $\int \frac{\cos x}{1+\sin x} dx$ 1

c) Use Simpson's rule with 3 ordinates to find an approximate value for

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{x} dx .$$ 3

Express your answer as a fraction.

d) Given the curve with equation $y = x^3 + 5x^2 + 3x - 9$, find the turning points and determine their nature. 3

e) Find the sum of geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{4096}$ 2

f) The curve $y = x^2 + ax + b$ has a turning point at (1,3). Find a and b. 2

g) If a and b are the first and last terms of an arithmetic series of $r + 2$ terms. Find the second and $(r+1)^{th}$ term, expressing your answers as single fractions. 3

h) Show that the point A(-1,5) is the reflection (image) of point B(3,-3) in the line $2y = x + 1$. 3

End of Section B.

Section C – Start a new booklet.

Question 3 (20 marks). **Marks**

a)

(i) Sketch on the same axes the graphs of $y = -\cos 2x$ and $y = \dfrac{x}{2}$ in the interval, $-\pi \le x \le \pi$ 3

(ii) Hence, find the number of solutions to the equation $-\cos 2x = \dfrac{x}{2}$, lying in the above interval. 2

b) A region R in the first quadrant is bounded by the y-axis, x-axis, the line $x = 3$ and the curve $y^2 = 4 - x$

(i) Draw a sketch showing the region R. 2

(ii) Calculate the area of region R. 3

(iii) Calculate the volume formed when R is rotated about the y-axis through one revolution. 3

c) An insurance policy pays the policy holder a percentage of his salary if he is unable to work. For the first month the payment is 100% of his salary, for the second month 97% and for the third month 94.15%. The percentages are calculated according to the formula:

$$P_{n+1} = aP_n + b$$

Where P_n is the percentage paid in month *n*, with *a* and *b* as constants.

(i) Find *a* and *b*. 2

(ii) Show that $P_3 = 100(0.95)^2 + 2(1 + 0.95)$ and that $P_n = 100(0.95)^{n-1} + 2(1 + 0.95 + ... + 0.95^{n-2})$, where $n \ge 2$ 2

(iii) Hence, given that the sequences of the percentages tends to a limit *P*, find the value of *P*. 3

End of Section C.

End of Examination.

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

1) a) $T_n = 12 - 4n$

$T_1 = 12 - 4(1)$

$T_1 = 8$

$T_2 = 12 - 4(2)$

$T_2 = 4$

$a = 8$

$d = T_2 - T_1$

$d = 4 - 8$

$d = -4$

b) i) $y = \cos 2x$

$y' = -2\sin 2x$

ii) $y = e^{x^2+1}$

$y' = 2xe^{x^2+1}$

iii) $y = \ln x^5$

$y = 5\ln x$

$y' = \frac{5}{x}$

iv) $y = \frac{\sin x}{x^2}$ $\quad u = \sin x \quad v = x^2$, $\quad u' = \cos x \quad v' = 2x$

$y' = \frac{vu' - uv'}{v^2}$

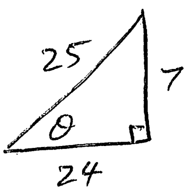
$y' = \frac{x^2\cos x - 2x\sin x}{x^4}$

$y' = \frac{x\cos x - 2\sin x}{x^3}$

c) $\frac{4\pi}{3} \times \frac{180}{\pi} = 240°$

d) $210 \times \frac{\pi}{180} = \frac{7\pi}{6}$

e)



$\cos\theta = \frac{24}{25}$

$\sin\theta = -\frac{7}{25}$

f) i) $\int x^{2n}\,dx$, where $n \neq -\frac{1}{2}$

$= \frac{x^{2n+1}}{2n+1} + C$

ii) $\int \sec^2(2x+3)\,dx$

$= \frac{1}{2}\tan(2x+3) + C$

g) $y = (x-3)(x^2+2)$

$y = x^3 + 2x - 3x^2 - 6$

$y' = 3x^2 + 2 - 6x$

when $x = 1$

$m_T = 3(1)^2 + 2 - 6(1)$

$m_T = -1$

h) $l = r\theta$

$l = 4\left(\frac{\pi}{3}\right)$

$l = \frac{4\pi}{3}$ cm

i) $y = x^2 - 3x + 2$

$y' = 2x - 3$

when $x = 0$

$m_T = -3$ $\quad$ point $(0, 2)$

$y - y_1 = m(x - x_1)$

$y - 2 = -3(x - 0)$

$y - 2 = -3x$

$y = -3x + 2$ or $3x + y - 2 = 0$

j) $A(-3,7)$, $B(2,19)$, $C(10,7)$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$AB = \sqrt{(2+3)^2 + (19-7)^2}$

$= 13$

$BC = \sqrt{(2-10)^2 + (19-7)^2}$

$= \sqrt{208}$

$AC = \sqrt{(10+3)^2 + (0)^2}$

$= 13$

since $AB = AC$

ΔABC is isosceles.

2007 Mathematics Assessment 2: **Solutions— Section B**

2. (a) The gradient function of a curve is $3x^2 - 5x + 1$ and the curve itself passes through (0, 3). Find the equation of the curve. [2]

Solution:
$$\frac{dy}{dx} = 3x^2 - 5x + 1,$$
$$y = x^3 - \frac{5x^2}{2} + x + 3.$$
$$\text{At } (0,\ 3) \quad 3 = c,$$
$$\therefore\ y = x^3 - \frac{5x^2}{2} + x + 3.$$

(b) Find:

i. $\int e^{3x+2} dx$, [1]

Solution: $\int e^{3x+2} dx = \dfrac{e^{3x+2}}{3} + c.$

ii. $\displaystyle\int \frac{\cos x}{1 + \sin x}\, dx.$ [1]

Solution: $\displaystyle\int \frac{\cos x}{1 + \sin x}\, dx = \ln(1 + \sin x) + c.$

(c) Use Simpson's rule with 3 ordinates to find an approximate solution for
$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{x}\, dx.$$
Express your answer as a fraction. [3]

Solution:
$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{x}\, dx \approx \tfrac{1}{3} \times \tfrac{1}{2}\left(2 + 4 \times 1 + \tfrac{2}{3}\right),$$
$$\approx \frac{1}{6} \times \frac{20}{3},$$
$$\approx \frac{10}{9}.$$

(d) Given the curve with equation $y = x^3 + 5x^2 + 3x - 9$, find the turning points and determine their nature. [3]

Solution:
$$\begin{aligned}\frac{dy}{dx} &= 3x^2 + 10x + 3,\\ &= (3x+1)(x+3),\\ &= 0 \quad \text{when } x = -\tfrac{1}{3},\ -3.\\ \frac{d^2y}{dx^2} &= 6x + 10,\\ &= 8 \quad \text{when } x = -\tfrac{1}{3},\\ &= -8 \quad \text{when } x = -3.\end{aligned}$$
$\therefore$ Maximum turning point at $(-3,\ 0)$.
Minimum turning point at $(-\frac{1}{3},\ -10\frac{16}{27})$ or $(-\frac{1}{3},\ -\frac{286}{27})$.

(e) Find the sum of the geometric series $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{4096}$. [2]

Solution:
$$\begin{aligned} a &= 1,\\ r &= \tfrac{1}{2},\\ \left(\tfrac{1}{2}\right)^{n-1} &= \left(\tfrac{1}{2}\right)^{12},\\ \therefore\ n &= 13.\\ \text{So sum } &= \frac{1-\left(\frac{1}{2}\right)^{13}}{1-\frac{1}{2}},\\ &= 2 \times \left(\frac{8192-1}{8192}\right),\\ &= \frac{8191}{4096}.\end{aligned}$$

(f) The curve $y = x^3 + ax + b$ has a turning point at $(1,\ 3)$.
Find a and b. [2]

Solution:
$$\begin{aligned} y &= x^3 + ax + b,\\ \text{i.e. } 3 &= 1 + a + b,\\ a + b &= 2.\\ \frac{dy}{dx} &= 2x + a,\\ \text{i.e. } 0 &= 2 + a,\\ a &= -2.\\ -2 + b &= 2,\\ b &= 4.\end{aligned}$$

(g) If a and b are the first and last terms of an arithmetic series of $r+2$ terms, find the second and $(r+1)^{\text{th}}$ terms, expressing your answers as single fractions. 3

Solution: $t_1 = a, \quad t_{r+2} = b,$

$$\begin{aligned}
i.e.\ b &= a + (r+2-1)d, \\
\therefore\ d &= \frac{b-a}{r+1}. \\
t_2 &= a + \frac{b-a}{r+1}, \\
&= \frac{ar+a+b-a}{r+1}, \\
&= \frac{ar+b}{r+1}. \\
t_{r+1} &= b - \frac{b-a}{r+1}, \\
&= \frac{br+b-b+a}{r+1}, \\
&= \frac{a+br}{r+1}.
\end{aligned}$$

(h) Show that the point $A(-1,\ 5)$ is the reflection (image) of point $B(3,\ -3)$ in the line $2y = x+1$. 3

Solution:

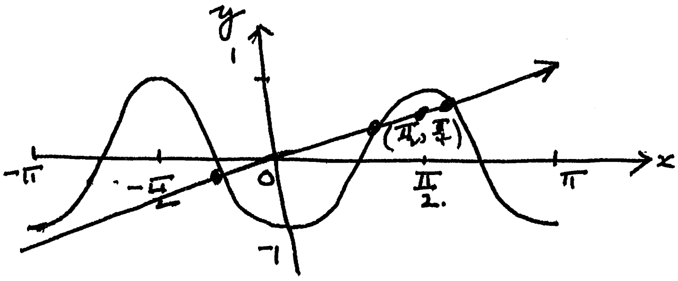
$$\begin{aligned}
\text{Slope of } 2y &= x+1 \text{ is } \tfrac{1}{2}. \\
\text{Slope of } AB &= \frac{5+3}{-1-3}, \\
&= -2.
\end{aligned}$$

$$\begin{aligned}
\text{Distance of } A \text{ from the line } &= \frac{|-1-2(5)+1|}{\sqrt{1+2^2}}, \\
&= \frac{10}{\sqrt{5}}. \\
\text{Distance of } B \text{ from the line } &= \frac{|3-2(-3)+1|}{\sqrt{1+4}}, \\
&= \frac{10}{\sqrt{5}}.
\end{aligned}$$

$\therefore$ The line is the perpendicular bisector of AB, *i.e.* A is the reflection of B in the line.

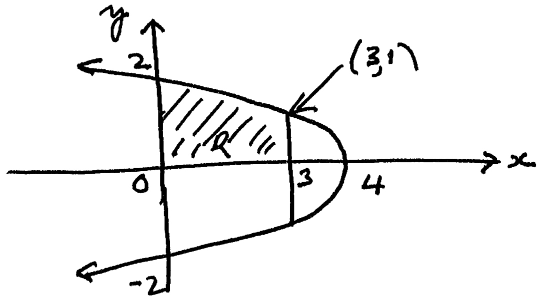
QUESTION 3.

(a) (i)



(ii) 3

(b) (i)



(ii) $\int_0^3 \sqrt{4-x}\, dx$

$= \int_0^3 (4-x)^{\frac{1}{2}} dx$

$= -\frac{2}{3}\left[(4-x)^{3/2}\right]_0^3$

$= -\frac{2}{3}\left(1^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$

$= -\frac{2}{3}(1-8)$

$= \frac{14}{3}\ u^2.$

(iii) $V = \pi\int_0^1 3^2\, dy + \pi\int_1^2 x^2\, dy.$

$= \pi\int_0^1 9\, dy + \pi\int_1^2 (4-y^2)^2 dy$

$= \pi\left[9y\right]_0^1 + \pi\int_1^2 (16 - 8y^2 + y^4)\, dy.$

$= 9\pi + \pi\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_1^2$

$= 9\pi + \pi\left[32 - \frac{64}{3} + \frac{32}{5} - \left(16 - \frac{8}{3} + \frac{1}{5}\right)\right]$

$= 9\pi + \pi\left[16 - \frac{56}{3} + \frac{31}{5}\right]$

$= 9\pi + \pi\ \frac{240 - 280 + 93}{15}$

$= 9\pi + \frac{53\pi}{15}$

$= \frac{188}{15}\pi\ u^3.$

c (i) $97 = 100a + b$ —①

$94.15 = 97a + b$ —②

① − ②

$2.85 = 3a$

$\boxed{a = 0.95}$

Substitute in ①

$97 = 100 \times 0.95 + b$

$97 = 95 + b.$

$\therefore \boxed{b = 2.}$

(ii) $P_2 = 100 \times 0.95 + 2.$

$P_3 = P_2 \times 0.95 + 2$

$= (100 \times 0.95 + 2)\,0.95 + 2.$

$= 100(0.95)^2 + 2 \times 0.95 + 2$

$= 100(0.95)^2 + 2(1 + 0.95)$

$P_4 = 100\,(0.95)^3 + 2(1 + 0.95 + 0.95^2)$

etc.

$P_n = 100(0.95)^{n-1} + 2\,(1 + 0.95 + 0.95^2 + \cdots + 0.95^{n-2})$

which is true for $n \geq 2$.

(iii) now $P_n = 100(0.95)^{n-1} + \dfrac{2(1 - 0.95^{n-1})}{1 - 0.95}.$

$P_n = 100\,(0.95)^{n-1} + \dfrac{2(1 - 0.95^{n-1})}{1/20}.$

$= 100\,(0.95)^{n-1} + 40\,(1 - 0.95^{n-1})$

now as $n \to \infty$ $0.95^{n-1} \to 0$ and $P_n \to \boxed{P = 40.}$